Existence and pathwise uniqueness to an SPDE driven by colored α-stable noise

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- Introduction
- Main results
- Sketch of proofs
- General result
- Future work

Introduction

We first consider

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + X_t(x)^\beta \dot{W}_t(x), \quad t > 0,\ x \in \mathbb{R},$$

where $\{W(dt, dx)\}$ is a space-time Gaussian white noise.

• (Nonnegative solution) $\beta = \frac{1}{2}$:
(1) The solution is the density of a SBM(Konno &Shiga(1988), Reimers(1989)).
(2) Pathwise uniqueness (PU): unknown.
(3) Xiong (2013): PU to SPDE for $\{Y_t\}$, $Y_t(x) := X_t(-\infty, x] := \int_{-\infty}^{x} X_t(y)dx$.
Dawson and Li (2012): PU to similar equation.

• Negative and positive solution (with $X_t(x)^\beta$ replaced by $|X_t(x)|^\beta$):
(1) $\beta > \frac{3}{4}$: Mytnik and Perkins (2011): PU holds
(2) $\beta \in (0, \frac{3}{4})$: non-uniqueness holds, Burdzy *et al.*(2010), Mueller *et al.*(2014)

•Colored noise:replaced W by $\tilde{W}(ds, dx) = \tilde{W}(ds, x)dx = dx \int_{y \in \mathbb{R}} \rho(y - x)W(ds, dy)$
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where $L(ds, dx)$ is a one sided α-stable white noise without negative jumps.

- $\alpha\beta = 1$:

(1) The solution is the density of a SBM with α-stable branching.

(2) PU: holds for $1 < \alpha < \sqrt{5} - 1$ (Y.& Zhou (2017)), unknown for rest.

(3) Fixed $t > 0$, $x \mapsto X_t(x)$ has continuous version ([MP03]) ($\star$)

Locally Hölder ([FMW10])($\star\star$) and Hölder continuous ([FMW11]).

It is almost sure multifractal (Mytnik and Wachtel (2015))

(4) He, Li and Y. (2014): PU to distribution-function-valued SPDE.

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replaced L by $\tilde{L}(ds, dx) := \tilde{L}(ds, x)dx = dx \int_{y\in\mathbb{R}} h(y - x)L(ds, dy)$.

Main results

Suppose that $\beta \geq 1 - 1/\alpha$ and $h \in L^{\theta}(\mathbb{R})^{+}$ for all $\theta > 0$.
$\mathcal{X} := \{f : \int_{\mathbb{R}} |f(x)|e^{-|x|}dx < \infty\}$, equipped with weak convergence topology.

Theorem 1 (Existence)

If $X_0 \in \mathcal{X}^+$ with $\int_{\mathbb{R}} X_0(x)^p e^{-|x|}dx < \infty$ for all $p < \alpha$, then (1) has a solution:

$$\sup_{t \in [0,T]} \mathbf{E}\Big\{ \int_{\mathbb{R}} X_t(x)^p e^{-|x|}dx \Big\} < \infty \text{ for all } p \in (0, \alpha).$$

Theorem 2 (PU)

If $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ are two solutions to (1) with $X_0 = Y_0$, $\mathbf{P}$-a.s.,

$$\mathbf{P}\Big\{ \int_{\mathbb{R}} |X_t(x) - Y_t(x)|dx = 0 \text{ for all } t \geq 0 \Big\} = 1.$$

Remark: PU for SDE holds with $\beta \geq 1 - 1/\alpha$ (Fu and Li (2010), Li and Mytnik (2011), and Li and Pu (2012)).

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Sketch of proof for Theorem 1

By (1),

$$X_t(x) = X_0(x) + \frac{1}{2}\int_0^t \Delta X_s(x)ds + \int_0^t \int_{\mathbb{R}} X_{s-}(x)^\beta h(y-x)L(ds,dy).$$

Inspired by Sturm (2003), we prove that SDE system has strong PU solution:

$$u_t^n(x^n) = u_0^n(x^n) + \frac{1}{2}\int_0^t \Delta^n u_s^n(x^n)ds + \int_0^t \int_{\mathbb{R}} [u_{s-}^n(x^n)]^\beta h^n(y,x^n)L(ds,dy), \quad (2)$$

where $x^n \in \frac{\mathbb{Z}}{n}$, $I_x^n := (x - \frac{1}{2n}, x + \frac{1}{2n}]$,

$$u_0^n(x) := n\int_{I_x^n} u_0(v)dv, \quad h^n(y,x) := n\int_{I_x^n} h(y-v)dv$$

and Δ^n is a discrete Laplacian operator defined by

$$\Delta^n f(z) := n^2[f(z+1/n) + f(z-1/n) - 2f(z)].$$

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$$(X_t,f) = (X_0,f) + \frac{1}{2}\int_0^t (X_s, \Delta f)ds + \int_0^t \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} X_{s-}(x)^\beta h(y-x)f(x)dx\Big] L(ds, d$$

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Sketch of proof for Theorem 2

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- $\phi_n(x) := \int_0^{|x|} dy \int_0^y \psi_n(z) dz$ with $a_n := \exp\{-n(n+1)/2\}$, $\int_{a_n}^{a_{n-1}} \psi_n(x) dx = 1$ and $0 \le \psi_n(x) \le 2n^{-1} x^{-1} 1_{(a_n, a_{n-1})}(x)$. Then $\phi_n'' \ge 0$.
- By Itô's formula,

$$\begin{aligned} \phi_n(\bar{X}_t^{\delta,r}(x)) &= -\frac{1}{2}\int_t^T \phi_n''(\bar{X}_s^{\delta,r}(x)) |\nabla \bar{X}_s^{\delta,r}(x)|^2 ds - \int_t^T \phi_n'(\bar{X}_s^{\delta,r}(x)) \nabla \bar{X}_s^{\delta,r}(x) dB_s \\ &\quad + \int_t^T ds \int_{\mathbb{R}} dy \int_0^\infty \mathcal{D}_n(\bar{X}_s^{\delta,r}(x), z\bar{H}_s^{\delta,r}(x,y)) \pi(dz) \\ &\quad + \int_{t-}^{T-} \int_0^\infty \int_{\mathbb{R}} D_n(\bar{X}_s^{\delta,r}(x), z\bar{H}_{s+}^{\delta,r}(x,y)) \widetilde{N}(\overleftarrow{ds}, dz, dy), \end{aligned}$$

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$$\begin{aligned} \phi_n(\bar{X}_t^{\delta,r}(x)) &= -\frac{1}{2}\int_t^T \phi_n''(\bar{X}_s^{\delta,r}(x)) |\nabla \bar{X}_s^{\delta,r}(x)|^2 ds - \int_t^T \phi_n'(\bar{X}_s^{\delta,r}(x)) \nabla \bar{X}_s^{\delta,r}(x) dB_s \\ &+ \int_t^T ds \int_{\mathbb{R}} dy \int_0^\infty \mathcal{D}_n(\bar{X}_s^{\delta,r}(x), z\bar{H}_s^{\delta,r}(x,y)) \pi(dz) \\ &+ \int_{t-}^{T-} \int_0^\infty \int_{\mathbb{R}} D_n(\bar{X}_s^{\delta,r}(x), z\bar{H}_{s+}^{\delta,r}(x,y)) \widetilde{N}(\overleftarrow{ds}, dz, dy), \end{aligned}$$

where $\mathcal{D}_n(y,z) := \phi_n(y+z) - \phi_n(y)$ and $D_n(y,z) := \phi_n(y+z) - \phi_n(y) - z\phi_n'(y)$.

Sketch of proof for Theorem 2

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Sketch of proof for Theorem 2

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- Then for $\theta, \gamma > 0$, $(\pi(dz) = cz^{-1-\alpha}dz)$,

$$\int_0^\infty \mathcal{D}_n(\bar{X}_s^r(x), z\bar{H}_s^r(x,y))\pi(dz)$$

$$= \int_0^{n^\gamma|\bar{X}_s^r(x)|^\theta} \mathcal{D}_n(\bar{X}_s^r(x), z\bar{H}_s^r(x,y))\pi(dz) + \int_{n^\gamma|\bar{X}_s^r(x)|^\theta}^\infty \mathcal{D}_n(\bar{X}_s^r(x), z\bar{H}_s(x,y))\pi(dz)$$

$$\leq Cn^{-1}|\bar{X}_s^r(x)|^{2\beta-1}h(y-\xi_s^r-x)^2 \int_0^{n^\gamma|\bar{X}_s^r(x)|^\theta} z^2\pi(dz)$$

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$$0 \le 2\beta - 1 + (2-\alpha)\theta < \alpha, \;\; 0 \le \beta + (1-\alpha)\theta < \alpha.$$

The above inequalities are equivalent to

$$\frac{1-2\beta}{2-\alpha} \le \theta \le \frac{\beta}{\alpha-1} \wedge \frac{\alpha+1-2\beta}{2-\alpha},$$

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- Letting $n \to \infty$,

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General result

Theorem

Suppose that

(C1): $G(0) \geq 0, H(0) = 0$, and the function $x \mapsto H(x)$ is nondecreasing.

(C2):

$$|G(x) - G(y)| \leq C|x - y|,$$
$$|H(x) - H(y)| \leq C|x - y|^{\beta}$$

for some constant $1 - 1/\alpha \leq \beta < 1$. Then for the following SPDE, Theorems 1 and 2 also hold:

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + G(X_t(x)) + H(X_{t-}(x))\dot{\tilde{L}}_t(x), \qquad t > 0,\ x \in \mathbb{R}^d.$$

Future work

- Let $h(x) = p_\gamma(x) := \frac{1}{\sqrt{2\pi\gamma}} e^{-x^2/(2\gamma)}$.

  Could we get PU to SPDE driven by stable white noise as $\gamma \to 0$?

Thanks!

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